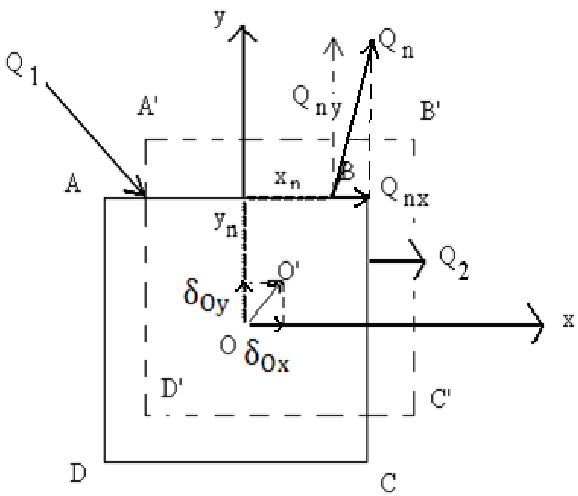
DEFLECTION BY VIRTUAL WORK METHOD

(A) Principle of virtual work for Rigid Bodies.

Consider a truly rigid body which is in static equilibrium under a system of forces Q_i.
Suppose this rigid body is TRANSLATED a small amount (say δ_0) by some other cause (which is independent of Q_i force system).
Since δ_0 is small all Q_i forces are assumed to maintain the same position and direction relative to the rigid body and to each other, and hence to maintain EQUILIBRIUM during this rigid translation:



$$\sum Q_{nX} \overset{\overset{+}{\rightarrow}}{=} 0 \qquad (10)$$

$$\sum Q_{nY} \overset{\uparrow +}{=} 0 \qquad (11)$$

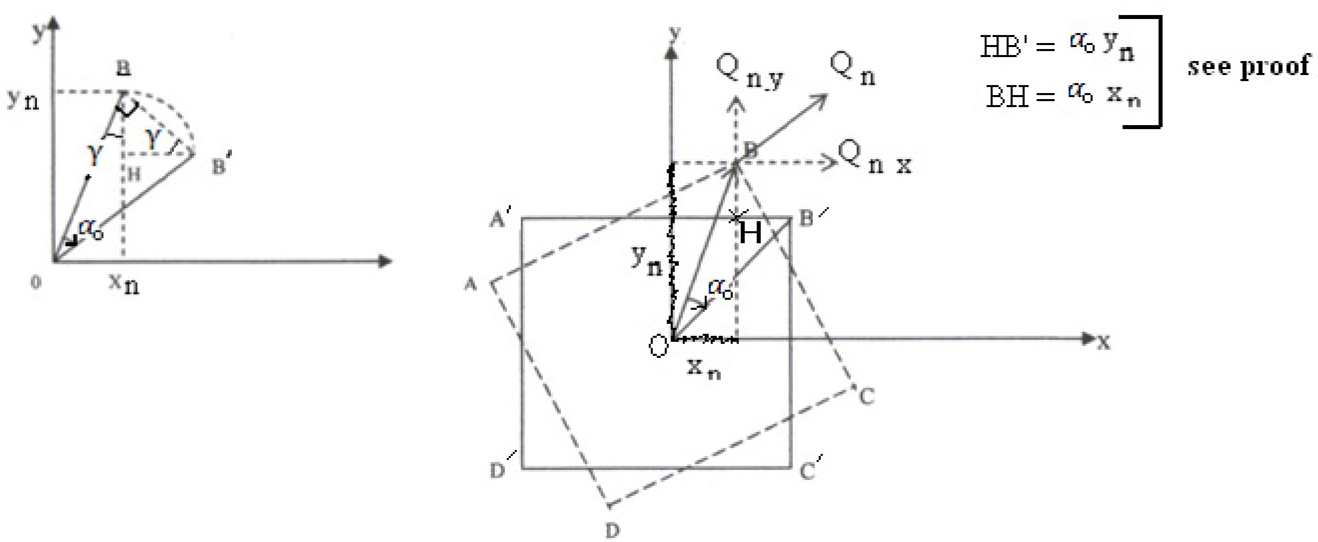
$$\sum M_0 = 0 \overset{\circlearrowleft +}{=} \sum(Q_{nx} * y_n) + (-Q_{ny} * x_n) \qquad (12)$$

Work done by the
Q_i forces $= W_Q = \sum(Q_{nx}\delta_{0x} + Q_{ny}\delta_{0y}) = (\delta_{0x})\cancel{\sum Q_{nx}} + (\delta_{0y})\cancel{\sum Q_{ny}}$ (13)

Zero

$$W_Q = 0 \qquad (14)$$

Similarly , if the rigid body is ROTATED a small amount $\propto_0$ by some other cause (which is independent of Q_i forces), then:



Since: α_0 is a small angle $\Rightarrow \widehat{OBB'} \approx 90^o$

Hence $\widehat{OBH} \equiv \gamma \cong \widehat{HB'B}$

We have:

$$OB = \sqrt{x_n^2 + y_n^2} \quad \text{And} \quad BB' \approx (OB)(\alpha_o^{rad}) = \sqrt{x_n^2 + y_n^2}\,(\alpha_o)$$

$$HB' = BB' * \cos(\gamma) = BB' * \frac{y_n}{OB} = \sqrt{x_n^2 + y_n^2}\,(\alpha_o) * \frac{y_n}{\sqrt{x_n^2+y_n^2}} = \propto_o y_n$$

$$BH = BB' * \sin(\propto) = BB' * \frac{x_n}{OB} = \sqrt{x_n^2 + y_n^2}\,(\alpha_o) * \frac{x_m}{\sqrt{x_n^2+y_n^2}} = \propto_o x_n$$

Now,

$$W_Q = \sum(Q_{nx} * HB' - Q_{ny} * BH) = \sum(Q_{nx} * \propto_o y_m - Q_{ny} * \propto_o x_n) \ldots\ldots(15)$$

$$W_Q = (\propto_o) \underbrace{\sum(Q_{nx}\, y_n - Q_{ny} x_n) = 0}_{\text{Zero (see Eq. 12)}}$$

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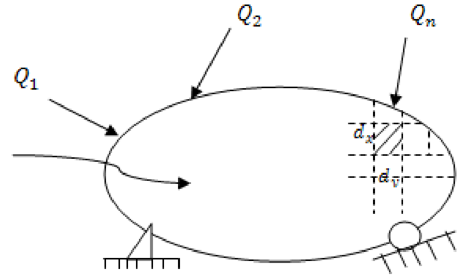
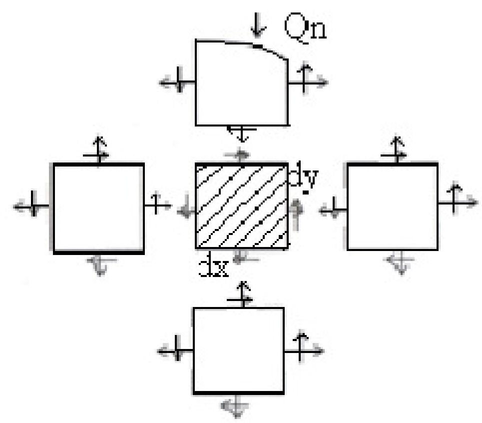
(B) Principal of Virtual Work for Deformable Bodies:



Step 1: This deformable body is in Equilibrium under the applied virtual forces Q_i and support reactions.

Step 2: Now , suppose the body is subjected to a small change in shape, caused by another load P system (which is independent
from Q_i system)

Step 3: Any INTERIOR particle, with dimensions d_x and d_y, will be displaced in its boundary and will have its virtual work to be canceled
by its adjacent particle(s).



Step 4:

Virtual work done on the EXTERIOR (or boundary)

Particle is given by:
$\mathrm{dw_S} = \mathrm{dw_R} + \mathrm{dw_d} \ldots\ldots\ldots\ldots\ldots (16)$

Where:

$\mathrm{dw_R}$ = due to RIGID body Translation & *Rotation* = 0 (*see Eq*. 14 & Eq. 15)

$\mathrm{dw_d}$ = due to deformable body

Hence: $\mathrm{dw_S} = \mathrm{dw_d} \Rightarrow \int \mathrm{dw_S} = \int \mathrm{dw_d}$ or $\boxed{\mathrm{w_s} = \mathrm{w_d}}$.........(17)

(C) **How to Evaluate W_S & W_d in Eq. (17) ?**

In Eq. (17), one has: $W_s = W_d$

Where W_S & W_d represents the External Virtual Work (VW),
and the Internal V.W., respectively

$W_s = Q * \delta$ = ("Virtual" Force or Moment) *("Real" Displacement, or Rotation) ……..(18)

The internal V.W. (=W_d) of deformation can be computed as:

$$W_d = (W_d)_{Axial} + (W_d)_{Bending} + (W_d)_{Torsion} + (W_d)_{Shear} \text{ ………..(19)}$$

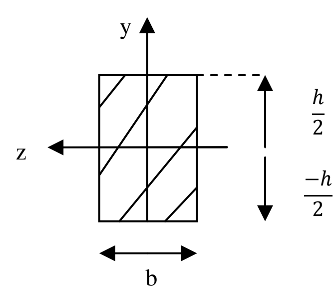
With:

$$(W_d)_{Axial} = (F_Q) * \left(\Delta L_{real} = \frac{F_p L}{AE} + \alpha_T * \Delta T * L + \Delta L_{Fabrication\ Error}\right) \ldots (19a)$$

$$(W_d)_{Bending} = \sum_{All\ members} \int_{s=0}^{L} \int_{y=\frac{-h}{2}}^{\frac{h}{2}} \underbrace{\underbrace{\left(\frac{M_Q y}{I}\right)}_{Stress} \underbrace{(bdy)}_{Area}}_{\text{"Virtual" Force}} * \underbrace{\underbrace{\left(\frac{M_p y}{IE}\right)}_{Strain} * \underbrace{(ds)}_{Length}}_{\text{"Real" Displacement}} = \sum \int_0^L M_Q * \frac{M_p ds}{EI} .. (19b)$$

Since:

$$\int_{y=-\frac{h}{2}}^{\frac{h}{2}} y^2 (bdy) = \left[\frac{by^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}} \equiv I$$



$$(W_d)_{\text{Torsion}} = (T_Q = \text{Virtual Torque}) * \left(\frac{T_P L}{JG} = \text{Real angle of twist}\right). \;.\;.\;.\;.\;. \quad (19c)$$

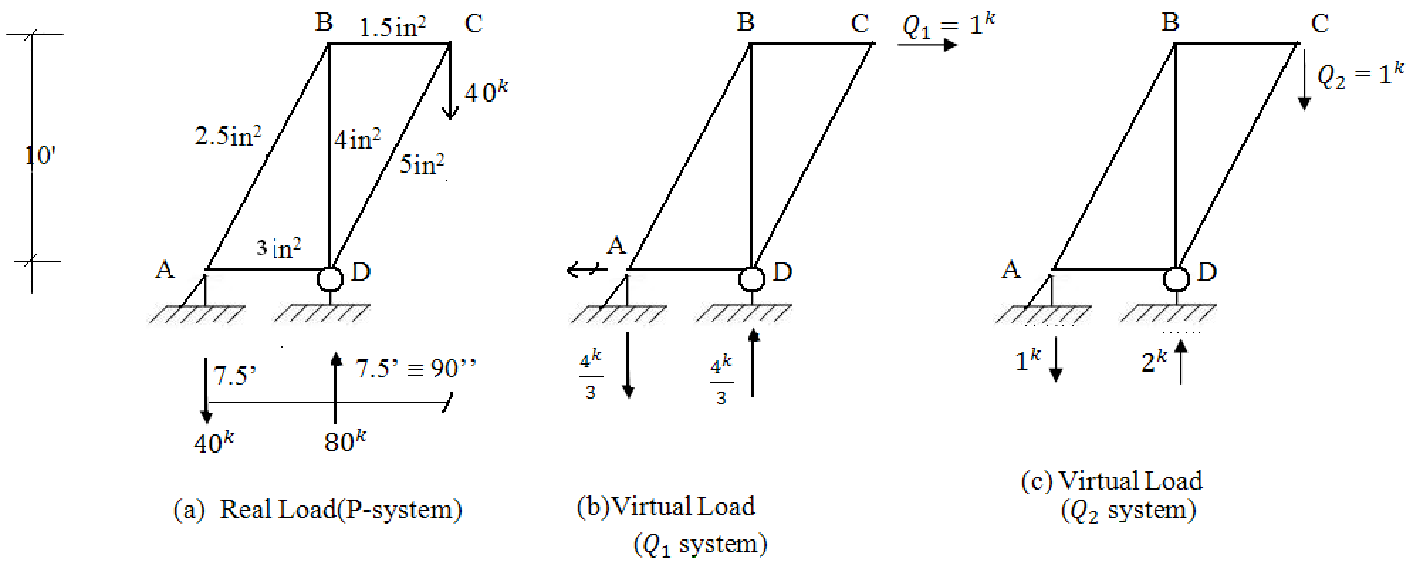
$$(W_d)_{\text{shear}} = k \int_{s=0}^{L} V_q \left(\frac{V_p}{AG}\right) ds \quad .\;. \quad (19d)$$

Where k=a constant, depending on the cross-sectional shape of the beam

<u>Important Notes</u>

- The nature of the applied "VIRTUAL" Load Force or Moment does depend on the "real" displacement Translation or Rotation that we want to compute (as will be demonstrated through many V.W. examples).
- In Eqs. (19a - 19d), the subscripts Q and P represents "Virtual" and "Real" loads, respectively.

Example 1: 2-D Truss with Applied Load



Units used = Kips, Inches
E = Young Modulus (K/in^2)

For the truss (shown in Fig. a), find the Horizontal & Vertical displacements at Joint C.

Solution for Example 1

- For finding the Horizontal displ. at Joint C (Δc_{horiz}); one applies Virtual Load Q_1 (Fig. b)
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- For finding the Vertical displ. at Joint C($\Delta c_{Vert.}$); one applies Virtual Load Q_2 (Fig. c)
- From V.W Formulas (see Eqs. 17; 19 a →19d) , one has ;

$$W_{s} + W_{settlement} = W_d \Longleftrightarrow (Q_1 = 1^{K} \rightarrow)(\Delta c_{horiz}^{??}) + \cancel{W_{settlement}} = \sum_{\text{All members}} F_{Q_i} \left(\frac{F_p L}{AE}\right)_i$$
.. (20)

0

$$(Q_2 = 1^{k} \downarrow)\ (\Delta c_{Vert.}^{??}) + \cancel{W_{settle}} = \sum_i F_{Q_{2i}} \left(\frac{F_p L}{AE}\right)_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
(21)

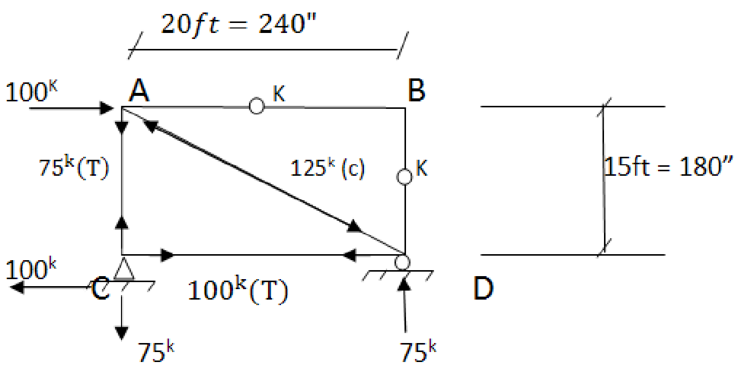
- From Static equilibrium , and applying Methods of Joints in figures (a + b + c), one gets:

Member	Length ≡ L	Area ≡ A	$\frac{L}{A}$	F_p	F_{Q_1}	F_{Q_2}	$F_{Q_1}\frac{F_pL}{A}$	$F_{Q_2}\frac{F_pL}{A}$
AB	150	2.5	60	50	5/3	5/4	5,000	3,750
BC	90	1.5	60	30	1	3/4	1/800	1,350
CD	150	5.0	30	-50	0	-5/4	0	1,875
BD	120	4.0	30	-40	-4/3	-1	1,600	1,200
AD	90	3.0	30	-30	0	-3/4	0	675
							$\sum_1 = 8{,}400$	$\sum_2 = 8{,}850$

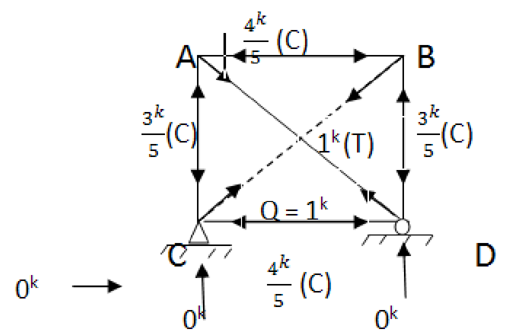
- Hence: $\Delta c_{horiz.} = \frac{+8400}{E}$ and $\Delta c_{Vert.} = \frac{8850}{E}$

Example 2: 2-D Truss with Applied Load

Find the "relative displacement" of joints C and B.
Area of each bar = 10 in^2 = A
Young Modulus = 30,000 $\frac{k}{in^2}$ = E



(a) Real Load (P-System)



(b) Virtual Load (Q-system)

$$W_S + W_{settlement} = \left(Q = 1^K \swarrow\!\nearrow\right)(\Delta_{CB}) + 0 = \sum_i F_{Q_i}\left(\frac{F_P L}{AE}\right)_i$$

$$= \sum\{(-\frac{3}{5} * 75^K * 180'') + (1^K * -125^K * 300'') + (-\frac{4^K}{5} * 100^K * 240'')\} * (\frac{1}{AE})$$

$$= -\frac{5400}{AE}$$

Thus: $\Delta_{CB} = \frac{-5400}{AE}$ (or $\Delta_{CB} = \frac{5400}{AE} \quad \swarrow\!\nearrow$)

Example 3: Plane Truss
(with Settlements, Fabrication Errors)
The truss shown is subjected to support settlements

At joints A $\left\{\begin{matrix} 3'' \rightarrow \\ 2'' \uparrow \end{matrix}\right.$ and B $\left\{\begin{matrix} 2'' \rightarrow \\ 3'' \downarrow \end{matrix}\right.$

What should be the length of member AD in order to have $(\Delta_D)_{Horizontal} = 0''$?

Given cross-sectional area of each bar = 20 in^2 = A
Young Modulus = 30,000 $\frac{K}{in^2}$ = E

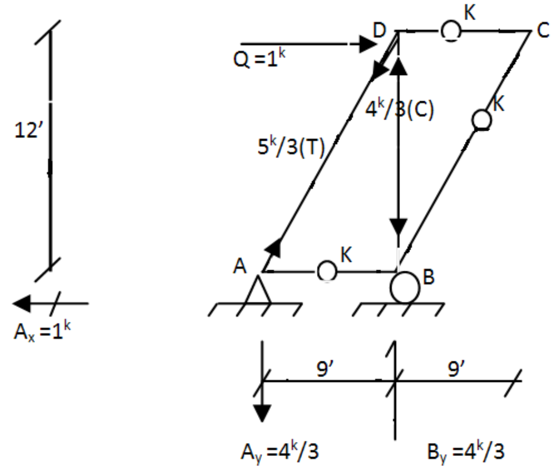


Fig. 1 : Virtual Load Applied

Solution

- Due to settlements' effects, horizontal displ. at joint D can be computed as:

$$W_s + W_{settlement} = \cancel{W_d} \text{ zero}$$

$$\left(Q = 1^k \rightarrow\right)\left(\Delta^1_{D_{Horiz.}}\right) + \left(+A_x = +1^k\right)(+3'') + \left(-A_y = -\frac{4^k}{3}\right)(2'') + \left(B_y = \frac{4^k}{3}\right)(-3'') = 0$$

Thus: $\left(\Delta^1_{D_{Horiz.}}\right) = +\left(\frac{11}{3}\right)^{in}$, or joint D moves <u>to the right.</u>

- In order to prevent the Horiz. displ. at joint D (due to settlement), length of member AD

has to be adjusted, so that joint D will move back <u>to the left</u>, by the same amount.

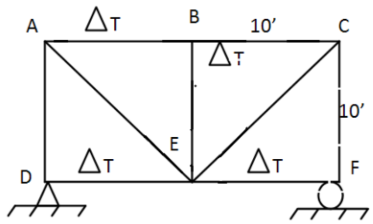
$$\left(W_s = 1^K * \frac{-11in}{3}\right) + \underbrace{\cancel{W_{settle}}}_{0} = \left(F_{Q_{AD}} = \frac{5^K}{3}\right)\left(\Delta L_{\substack{\text{Fabrication} \\ \text{Error for AD}}} ??\right) \rightarrow (\Delta L_{Fab.Err})_{AD} = -\frac{11''}{5}$$

Example 4: Plane Truss with Temperature, Pre-strain (or Fabrication Error), Support settlements … effects

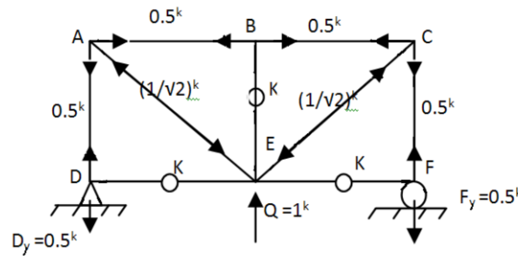
Given: $E = 30{,}000\ \frac{k}{in^2}$

$\alpha = 6.5 * 10^{-6} \frac{in}{in°F}$

Find: Vertical displ. at joint E.



(a) Real "load" P-System



(b) Virtual Load Q-System

(A) Due to temperature of bars AB, BC, DE & EF is raised by 100°F

$$W_{s} + W_{settlement} = W_d$$

$$(Q = 1^k \uparrow)(\Delta_{E_{vert.}}) + 0 = \sum_i F_{Q_i}\,(\alpha \Delta T * L)_i \text{ where } \left\{ \begin{array}{c} i = \text{bars AB, BC, DE, EF} \\ (F_Q)_{AB} = 0.5^k = (F_Q)_{BC} \\ (F_Q)_{DE} = 0^k = (F_Q)_{EF} \end{array} \right\}$$

So: $\Delta_{E_{vert.}} = +6.5 * 10^{-3}$ft (or moves upward)

(B) Due to Pre-strain (or Fabrication Error)

Bars AB, BC, DE, EF, AD, BE, CF are 2" too short.
Bars AE, CE are $\sqrt{2}$=1.414" too short

$$W_S + W_{settle} = W_d$$

$(Q = 1^K \uparrow)(\Delta_{E_{vert}}) + 0 = \Sigma(F_{Q_i})(\Delta L_{P_i})$ where i=AB, …, CF; AE, CE

$$= \underbrace{(0.5^k)(-2'')}_{AB} + \ldots + \underbrace{(0.5^k)(-2'')}_{CF} + \underbrace{\left(\frac{-1^k}{\sqrt{2}}\right)\left(-\sqrt{2}''\right)}_{AE} + \underbrace{\left(\frac{-1^k}{\sqrt{2}}\right)\left(-\sqrt{2}''\right)}_{CE}$$

So: $\Delta_{E_{vert}} = -2''$ (or moves downward)

(C) <u>Due to settlements at Joint D</u> $\begin{cases} \mathbf{2}'' \leftarrow \\ \mathbf{1}'' \downarrow \end{cases}$ **, <u>and at Joint F</u>** $\begin{cases} \mathbf{3}'' \rightarrow \\ \mathbf{4}'' \downarrow \end{cases}$

$W_S \quad + \quad W_{settle} \quad = \quad W_d$

$(Q = 1^K \uparrow)(\Delta_E)_{vert} + \left(D_y = -0.5^k\right)\left(-1''\right) + (Fy = -0.5^k)(-4'') = 0$

So: $\Delta_{E_{vert}} = -2.5''$ (or moves <u>downward</u>)

(D) **<u>Due to all, combined effects</u>**

$$(\Delta_E)_{vert} = +\left(6.5 * 10^{-3}\text{ft} * \frac{12''}{\text{ft}}\right) + \left(-2'' + (-2.5'')\right) = -4.42''$$

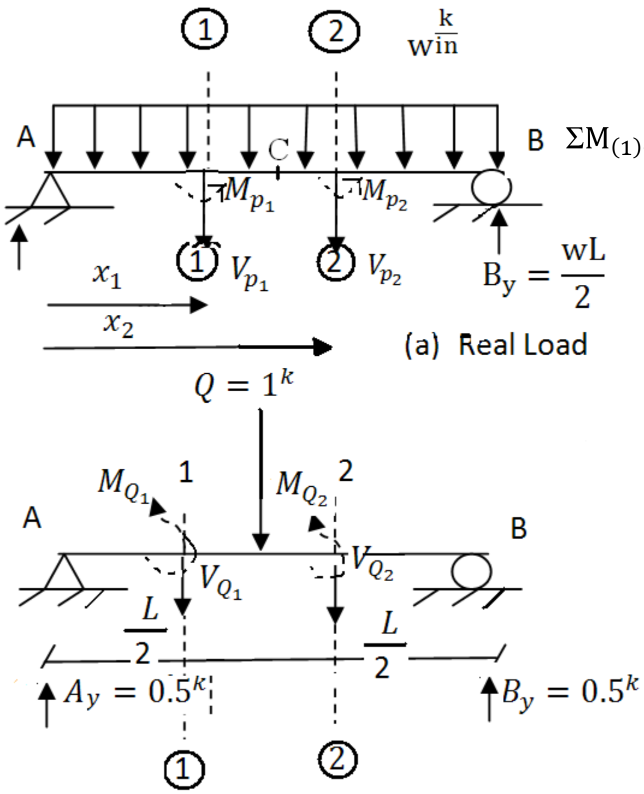
Example 5: V.W on Beam

Find the vertical deflection Δ_c at mid-point of the beam?

- Use left-side of section (1)-(1) as free Body Diagram (FBD)

$$\Sigma M_{(1)-(1)} = 0 = Mp_1 - \left(Ay = \frac{wL}{2}\right)(x_1) + (wx_1)\left(\frac{x_1}{2}\right)$$

Hence: $Mp_1 = \frac{wLx_1}{2} - \frac{wx_1^2}{2}$ for $0 \leq x_1 \leq \frac{L}{2}$



(a) Real Load

(b) Virtual Load

- Similarly:

$$Mp_2 = \frac{wLx_2}{2} - \frac{wx_2^2}{2}$$

for $\frac{L}{2} \leq x_2 \leq L$

- Use left-side of section (1)-(1) as FBD:

$$\Sigma M_{(1)-(1)} = 0 \quad = M_{Q_1} - (Ay = 0.5^K)(x_1)$$

So: $M_{Q_1} = 0.5x_1$ for $0 \leq x_1 \leq \frac{L}{2}$

- Use left-side of section (2)-(2) as FBD:

$$\Sigma M_{(2)-(2)} = 0 \quad = M_{Q_2} - (Ay = 0.5^K)(x_2) + (1^k)\left(x_2 - \frac{L}{2}\right)$$

So: $M_{Q_2} = -0.5x_2 + \frac{L}{2}$ For $\frac{L}{2} \leq x_2 \leq L$

Now, applying V.W. formula (see Eq. 19 b)

$$W_S + W_{settle} = \int M_Q * \frac{M_p}{EI}\, dx$$

$$(Q = 1^k \downarrow)(\Delta_c??) + 0 = \int_{x_1=0}^{\frac{L}{2}} M_{Q_1} * \frac{M_{P_1}}{EI}\, dx_1 + \int_{x_2=\frac{L}{2}}^{L} M_{Q_2} * \frac{M_{P_2}}{EI}\, dx_2$$

Or: $\Delta_c = \int_{x_1=0}^{L/2} (0.5x_1) * \frac{\left(\frac{wLx_1}{2} - \frac{wx_1^2}{2}\right)}{EI} dx_1 + \int_{x_2=\frac{L}{2}}^{L} \left(-0.5x_2 + \frac{L}{2}\right) * \frac{\left(\frac{wLx_2}{2} - \frac{wx_2^2}{2}\right)}{EI} dx_2$

Hence: $\Delta_c = \frac{+5wL^4}{384EI}$ (or point C will move <u>downward</u>, as we have assumed for direction of Q)

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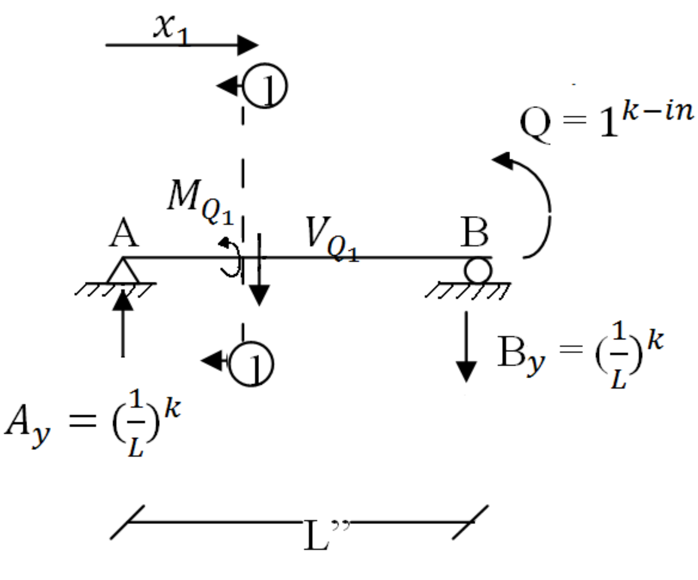
Example 6: More on V.W. for Beam



Virtual Load (Q-System)

For the same beam (Shown in Example 5), find the rotation (θ_B) at Joint B

$$\sum M_{(1)-(1)} = 0 \quad = M_{Q_1} - (Ay = \frac{1}{L})^{kip}(x_1)$$

Hence: $M_{Q_1} = \frac{x_1}{L}$ for $0 \le x_1 \le L$

Applying V.W. formula (See Eq. 19b):

$$W_S \quad + \quad W_{settle.} \quad = \quad W_d$$

$$(Q = 1^{k-in})(\theta_B) + 0 = \int M_Q \frac{M_p}{EI} dx$$

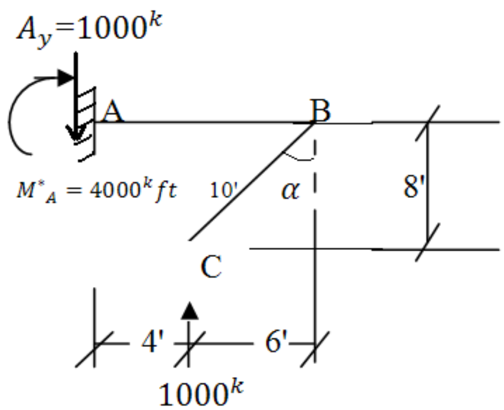
Or $$\boxed{\theta_B = \int_{x_1=0}^{L} (\frac{x_1}{L}) * \frac{(\frac{wLx_1}{2} - \frac{wx_1^2}{2})}{EI} dx_1}$$

Note:
The M_{p_1} computed earlier (see Example 5) can also be used for entire beam (or $0 \le x_1 \le L$)

Example 7: V.W. on Frame

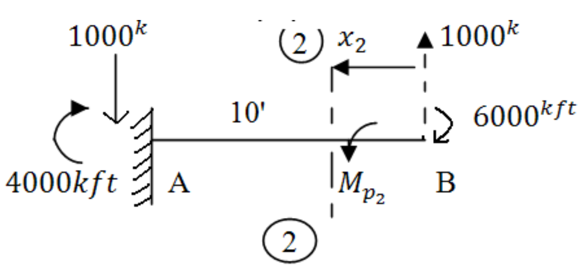
Given: $E = 30{,}000^{k/in^2}$; $I = 288\ in^4$

Find: vertical deflection at point C? Rotation at point C ?



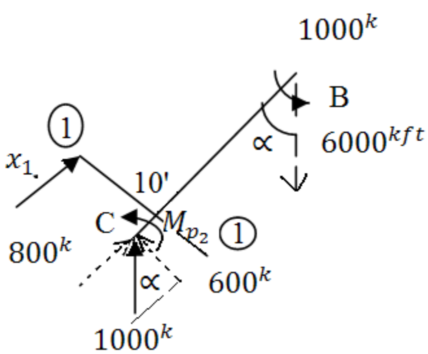
(a) Real Load, P-System

(b) Free body Diagrams (for P-system)



$\sum M_{(2)-(2)} = 0 \;\; = M_{p_2} + (1000^k)(x_2) - 600^{kft}$

So:
$M_{p_2} = 6000^{kft} - 1000x_2$
$0' \le x_2 \le 10'$



$\sum M_{(1)-(1)} = 0 \;\; = M_{p_1} - (600^k)(x_1)$

So:
$M_{p_1} = 600x_1$
$0' \le x_1 \le 10'$

Note: To find vertical deflection at point C, we could have also applied a "Virtual" vertical load $Q_1 = 1000^k$ at point C.
Hence, Fig. (b) can also be used for Q_1-system !

To find vertical deflection (Δ_c) at joint C

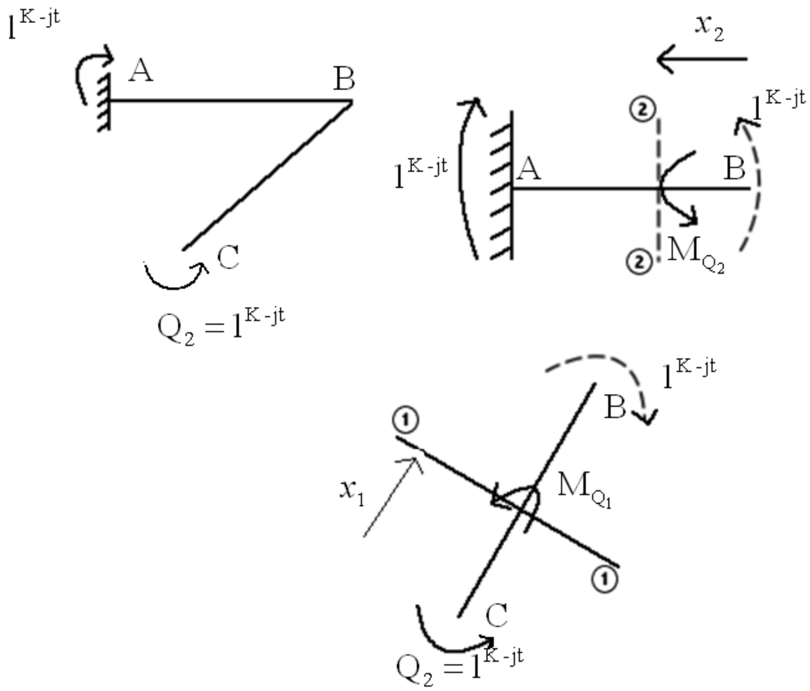
$$W_s \quad + \quad W_{settle} \quad = \quad W_d$$

$$(Q_1 = 1000^K)(\Delta^{ft}{}_c) \quad + \quad 0^{Kft} \qquad = \int M_Q * \frac{M_p dx}{EI}$$

Or
$$(1000)\,\Delta_c{}^{ft} = \int_{x_1=0}^{10} M_{Q_1} * \frac{M_{p_1}}{EI} dx_1 + \int_{x_2=0}^{10} M_{Q_2} * \frac{M_{p_2}}{EI} dx_2$$
$$= \int_0^{10} \frac{(M_{p_1})^2}{EI} dx_1 + \int_0^{10} \frac{(M_{p_2})^2}{EI} dx_2$$

$$(1000)\Delta_c{}^{ft} = \int_0^{10} \frac{(600x_1)^2}{EI} dx_1 + \int_0^{10} \frac{(6000^{kft} - 1000_{x_2})^2}{EI} dx_2$$

Thus:
$$\boxed{\Delta_c{}^{ft} = \left(\frac{1}{1000}\right) \left\{ \int_0^{10} \frac{(600x_1)^2}{EI} dx_1 + \int_0^{10} \frac{(6000 - 1000_{x_2})^2}{EI} dx_2 \right\}}$$



$\sum M_{(2)-(2)} = 0 = M_{Q_2} + 1^{K-ft}$

So:

$\boxed{M_{Q_2} = -1^{Kft}}$

Valid for $0' \leq x_2 \leq 10'$

$$\sum M_{(1)-(1)} = 0$$
$$= M_{Q_1} + 1^{Kft}$$

So:

$\boxed{M_{Q_1} = -1^{Kft}}$

Valid for $0' \leq x_1 \leq 10'$

(C)- Virtual Load (Q_2-System) and FBD of members AB, BC.

To find ROTATION (θ_c) at joint C

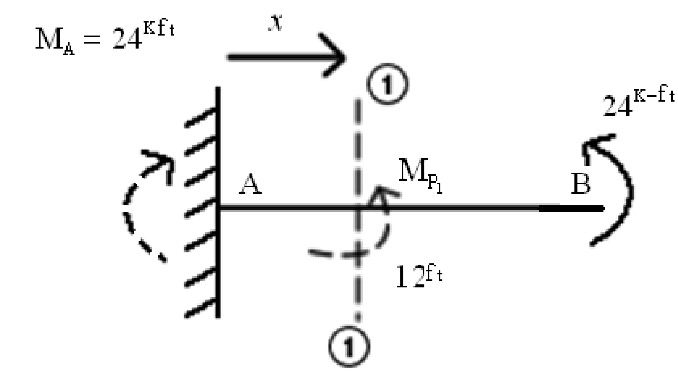
$$W_s \quad + \quad W_{settle} \quad = \quad W_d$$

$$\left(Q_2 = 1^{Kft} \circlearrowright\right)(\theta_C) + \; 0^{Kft} \qquad = \int M_Q * \frac{M_p dx}{EI}$$

Or: $\theta_C = \int_{x_1=0'}^{10'} M_{Q_1} * \frac{M_{P_1} dx_1}{EI} + \int_{x_2=0'}^{10'} M_{Q_2} * \frac{M_{P_2} dx_2}{EI}$

Or: $\boxed{\theta_C = \int_0^{10} (-1^{Kft}) * \frac{(600x_1)}{EI} dx_1 + \int_0^{10} (-1^{Kft}) * \frac{(6000^{Kft} - 1000x_2)}{EI} dx_2}$

<u>Example 8</u>: V.W. for a Determinate Beam
Find the vertical deflection of joint B.



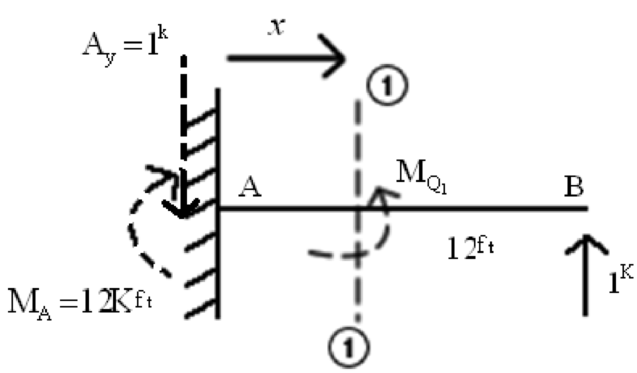
(a) Real Load (Psystem)

$$W_s = W_d$$

$$\left(1^k\right)\left(\delta_{B_V}\right) + \left(W_{settle} = 0\right) = \int \frac{M_{Q_1} M_{P_1} dx}{EI} \quad (8.1)$$

Use left side of cut section 1)-1) as FBD, then:

$$\sum M_1 = 0 = M_{P_1} - 24^{kft} \Rightarrow M_{P_1} = 24^{kft} \quad (8.2)$$



Use left side of the cut section 1)-1) as FBD, then:

$$\sum M_1 = 0 = M_{Q_1} + \left(1^k\right)(x) - 12^{kft}$$

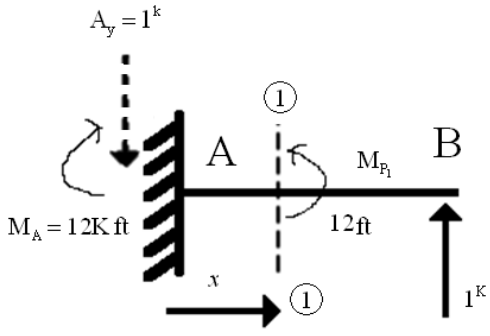
Hence: $M_{Q_1} = 12 - x$ (8.3)

Substitute Eqs. (8.2, 8.3) into Eq. (8.1), one gets

$$\delta_{B_V} = \int \frac{(12-x)(24)(dx)}{EI} = \ldots$$

(b)- Virtual Load (Q-System)

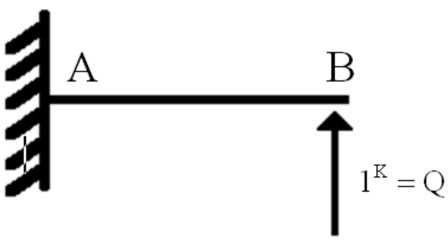
Example 9: V.W. for Determinate Beam
Find the vertical deflection at joint B? (Refer to example 8)



(*a*) Real (P) Load

$$W_s = W_d$$

$$\left(Q = 1^k \uparrow\right)\left(\delta_{B_V}\right) + \left(W_{settle} = 0\right) \int \frac{\left(M_{q_1}\left(M_{P_1} = M_{q_1}\right)\right.}{EI} d\boldsymbol{x}$$

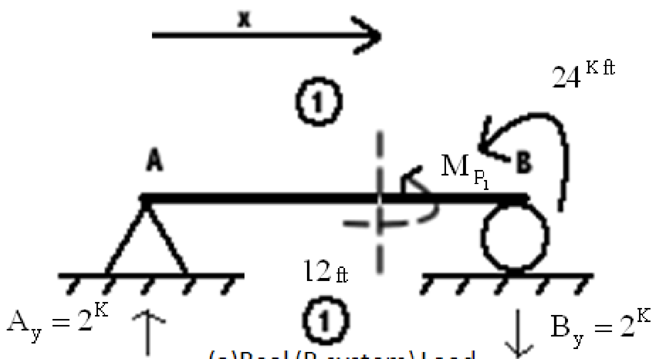


(*b*) Virtual (Q) Load

$$\delta_{B_V} = \int \frac{\left(M_{q_1}\right)^2 dx}{EI} = \int_0^{12} \frac{(12-x)^2}{EI} dx = \cdots$$

Example 10: V.W. for simply support, Determinate Beam
Find the ROTATION at joint A.

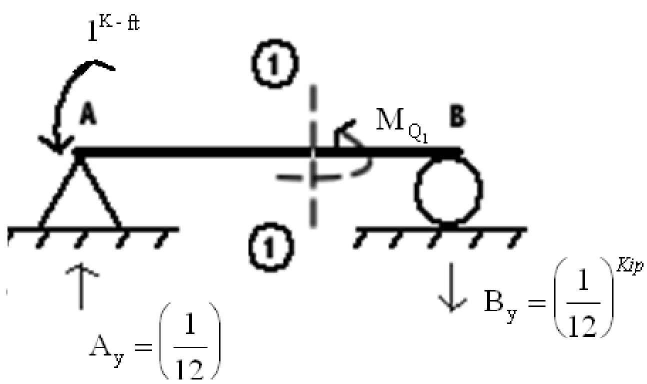
Use left side of cut section 1)-1) as FBD;



(a)Real (P-system) Load

$\sum M_{(1)} = 0 \quad = M_{P_1} - (2^K)(x) \Longrightarrow M_{P_1} = 2x..(10.1)$

$\sum M_{(1)} = 0 \quad = M_{Q_1} + 1^{kft} - (\frac{1}{12})(x)$



(b)Virtual (Q-system) Load

Hence: $M_{Q_1} = \frac{x}{12} - 1$ ………….(10.2)

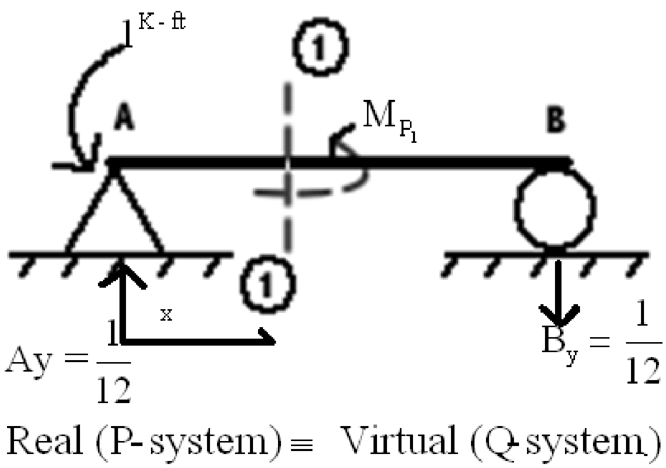
Use V.W:

$$W_s + W_{settle} = W_d$$

$$(Q{=}1^{Kft}\ \curvearrowleft)(\theta_A) + 0 = \int \frac{M_{Q_1} M_{P_1}}{EI} dx$$

$$\theta_A = \int_0^{12} (\frac{x}{12} - 1)(2x) \frac{dx}{EI} = \cdots$$

Example 11: V.W. for Simply Support, determine beam.
Find the rotation at Joint A.



Use left side of cut section (1)-(1) as FBD;

$$\sum M_1 = 0 = M_{P_1} + 1^{kft} - \left(\frac{1}{12}\right)(x)$$

Hence: $M_{P_1} = \frac{x}{12} - 1 \equiv M_{Q_1}$

Apply VW:

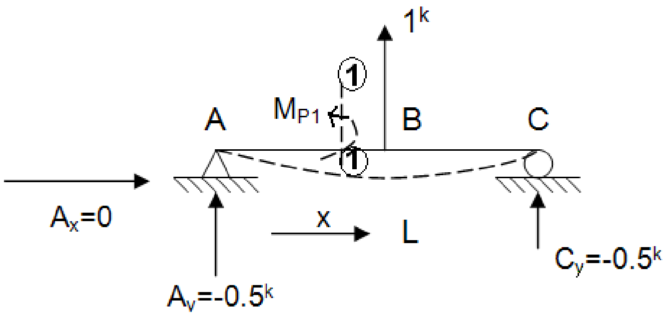
$$W_s + (W_{settle} = 0) = W_d$$

$$(Q=1^{kft}\ \curvearrowleft)(\theta_A) + 0 = \int M_{Q_1} M_{P_1} \frac{dx}{EI}$$

$$\theta_A = \int_0^{12} (\frac{x}{12} - 1)^2 \frac{dx}{EI} = \cdots$$

Example 12

Find the vertical displacement at joint B?



Use left-side of cut section (1)-(1) as FBD.

$$\sum M_{cut(1)} = 0 = M_{P_1} - (A_y)(x)$$

So: $M_{P_1} = A_y x = (-0.5)x \equiv M_{Q_1}$

Real (P-System) ≡ Virtual (Q- System)

Apply V.W.

$$W_s + W_{settle} = W_d$$

$$(Q=1^K \uparrow)(\delta_{BV}) + 0 = \int M_{Q_1} M_{P_1} \frac{dx}{EI}$$

$$\delta_{BV} = \int_0^L (-0.5\,x)^2 \frac{dx}{EI} = \ \ldots$$